Chapter 4: Integration

In this chapter, we develop the theory of integration of complex-valued functions of a complex-variable. Integrals will be defined over suitable curves in the complex plane. The theory of integration is a surprisingly powerful tool in the study of analytic functions.

The content of this chapter is essentially a characterization of analytic functions. Roughly speaking, we will prove the following theorem:

Let D be a domain and $f: D \longrightarrow \mathbb{C}$ a function. The following are equivalent:

(1) f is analytic on D;

(2) For all $n \in \mathbb{N}$, $f^{(n)}$ exists and is analytic on D;

(3) In each "simply connected" subdomain S of D, there is an analytic function $F: S \longrightarrow \mathbb{C}$ such that $F' = f$ on S.

(4) f is continuous on D and

$$\int f(z)\, dz = 0$$

over every contour C lying in any "simply connected" subdomain.

(5) If C is a simple closed contour in D and z_0 is interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Additionally, as an application of the theory, we will prove:

- Every bounded entire function is constant
- Every polynomial of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Derivatives of Functions of a Real-Variable

To define integrals of a complex-valued function of a complex variable, we need to understand how to differentiate a function of a real-variable

$$w : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$$

where I is an interval in $\mathbb{R}$.

Definition Let $I \subseteq \mathbb{R}$ be an interval and $w : I \to \mathbb{R}$ a function. Writing $w(t) = u(t) + iv(t)$, we define the derivative of w to be

$$w'(t) = u'(t) + i\, v'(t)$$

provided that u' and v' exist. In that case, w is differentiable.

Some rules for differentiation remain valid:

Proposition Suppose $w(t) = u(t) + i\, v(t)$ and $W(t) = U(t) + iV(t)$ are differentiable. Then

(1) $(w(t) + W(t))' = w'(t) + W'(t)$

(2) $(w(t)\, W(t))' = w'(t)\, W(t) + W'(t)\, w(t)$.

Note: there may be others.

Proof.

(1) $(w(t) + W(t))' = (u + U + i(v + V))' = (u+U)' + i(v+V)'$

$$= u' + U' + i(v' + V')$$
$$= (u' + iv') + (U' + iV')$$
$$= w'(t) + W'(t).$$

(2) $(wW)' = ((u+iv)(U+iV))'$

$$= (uU - vV + i(uV + vU))'$$
$$= (uU - vV)' + i(uV + vU)'$$
$$= u'U + U'u - (v'V + V'v) + i(u'V + V'u + v'U + U'v).$$

$$w'W = (u' + iv')(U + iV) = u'U - v'V + i(u'V + v'U)$$
$$W'w = (U' + iV')(u + iv) = U'u - V'v + i(U'v + V'u)$$

compare

$\boxed{\text{Example}}$ We will frequently encounter the function

$$w(t) = e^{z_0 t}, \quad z_0 \in \mathbb{C}, \quad t \in [a,b].$$

We compute $w'(t)$ for use later:

Decompose w into real/imaginary parts: (write $z_0 = x_0 + iy_0$)

$$w(t) = e^{z_0 t} = e^{x_0 t + ity_0}$$
$$= e^{x_0 t} \cdot e^{ity_0}$$

$$= e^{x_0 t} \cos y_0 t + e^{x_0 t} i \sin y_0 t.$$

Then

$$w'(t) = x_0 e^{x_0 t} \cos y_0 t - y_0 e^{x_0 t} \sin y_0 t + i(x_0 e^{x_0 t} \sin y_0 t + y_0 e^{x_0 t} \cos y_0 t)$$
$$= x_0 e^{x_0 t}(\cos y_0 t + i \sin y_0 t) + iy_0 e^{x_0 t}(\cos y_0 t + i \sin y_0 t)$$
$$= e^{x_0 t}(x_0 + iy_0) e^{iy_0 t}$$
$$= z_0 e^{x_0 t + iy_0 t} = z_0 e^{z_0 t}.$$

To summarize $\boxed{\dfrac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}.}$

Integral of a Function $w : I \subseteq \mathbb{R} \to \mathbb{C}$

Definition (Definite Integral of $w : I \subseteq \mathbb{R} \to \mathbb{C}$) Suppose that $w(t) = u(t) + i v(t)$ where $u(t), v(t)$ are real-valued functions of a real variable defined on an interval $[a,b]$. The definite integral of w over $[a,b]$ is defined by

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

provided that the integrals of u and v exist. //

Improper integrals over an unbounded interval are defined similarly.

Example To illustrate the definition, we integrate $w(t) = e^{it}$ on $[0, \pi]$.

$$\int_0^\pi e^{it}\, dt = \int_0^\pi \cos t\, dt + i \int_0^\pi \sin t\, dt$$

$$= 0 + 2i$$

$$= 2i.$$

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Definition (Piecewise continuity) A function $u : I \subseteq \mathbb{R} \to \mathbb{R}$ is piecewise continuous on I if it is continuous on I except at a finite number of points at which it is discontinuous, but has one sided limits. A function $w(t) = u(t) + iv(t)$ is piecewise continuous if both u and v are. //

Note: The existence of the integrals

$$\int_a^b u(t)\,dt \quad \text{and} \quad \int_a^b v(t)\,dt$$

is ensured when w is piecewise continuous on $[a,b]$.

$\boxed{\text{Proposition (Properties of the integral of } w:[a,b]\to\mathbb{C})}$ Suppose that $w(t)$ and $W(t)$ are piecewise continuous on $[a,b]$. Then

(1) $$\int_a^b z_0\, w(t)\,dt = z_0 \int_a^b w(t)\,dt \quad \text{for any } z_0 \in \mathbb{C};$$

(2) $$\int_a^b w(t) + W(t)\,dt = \int_a^b w(t)\,dt + \int_a^b W(t)\,dt\;;$$

(3) $$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt\,, \text{ any } c \in [a,b];$$

(4) $$\int_a^b w(t)\,dt = -\int_b^a w(t)\,dt\,.$$

Proof. All follow from properties of ordinary integrals. ∎

Proposition (Extension of Fundamental Theorem of Calc.) Suppose that $w(t) = u(t) + i\,v(t)$ is continuous on $[a,b]$ and $W(t) = U(t) + i\,V(t)$ is differentiable such that $W'(t) = w(t)$ on $[a,b]$. Then

$$\int_a^b w(t)\,dt = W(b) - W(a).$$

Proof. Assume $W'(t) = w$. This means $U' = u$ and $V' = v$. Hence,

$$\begin{aligned}\int_a^b w(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \\ &= U(b) - U(a) + i\big(V(b) - V(a)\big) \quad (\text{by FTC}) \\ &= U(b) + iV(b) - \big(U(a) + iV(a)\big) \\ &= W(b) - W(a).\end{aligned}$$

This proves the claim. ■

Example We use the proposition to integrate e^{it} on $[0,\pi]$.

Notice that $\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{i}{i}e^{it} = e^{it}$. By the theorem

$$\begin{aligned}\int_0^\pi e^{it}\,dt = \left[\tfrac{1}{i}e^{it}\right]_0^\pi &= \tfrac{1}{i}e^{i\pi} - \tfrac{1}{i}e^{0} \\ &= \tfrac{1}{i}\left(e^{i\pi} - 1\right) \\ &= \tfrac{1}{i}(-1 - 1) = -\tfrac{2}{i} = 2i.\end{aligned}$$

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Contours

So far, we have only defined the integral of a complex-valued function of a real variable over an interval. Integrals of complex-valued functions of a complex variable are defined on suitable curves in the complex plane, called contours.

Definition (arcs)

(1) An arc is a collection of points

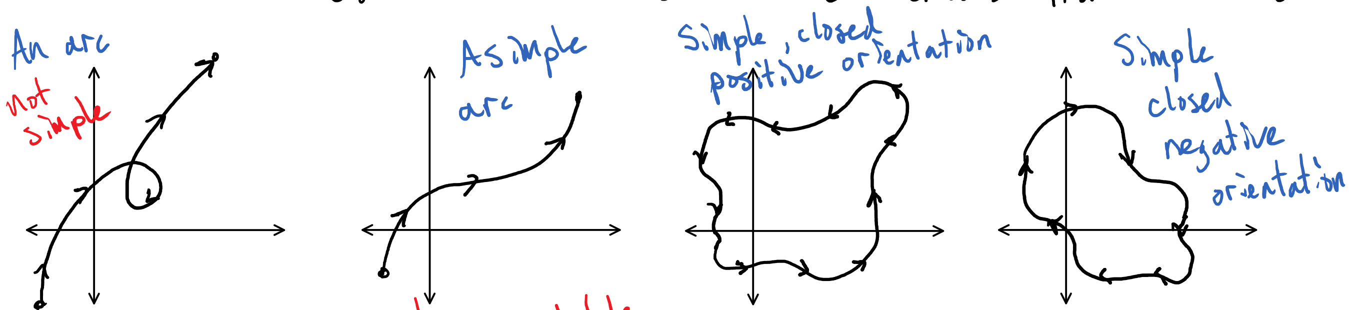
$$C = \{ z(t) : t \in [a,b] \}$$

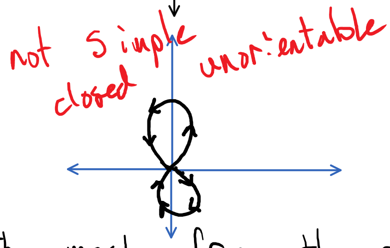
where $z(t) = x(t) + i\,y(t)$ and $x, y : [a,b] \longrightarrow \mathbb{R}$ are continuous functions. The function $z(t)$ is called a parameterization of C.

(2) An arc C is called simple or a Jordan arc if it does not cross itself: $z(t_1) = z(t_2) \Rightarrow t_1 = t_2$.

(3) If C is simple except for the fact that $z(a) = z(b)$, then C is called a simple closed curve or Jordan curve.

(4) A simple closed curve is positively oriented if it is traversed counter-clockwise as t increases from a to b.



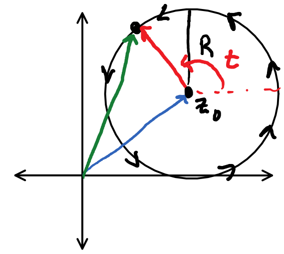


Example The most frequently encountered arcs and curves are line segments and circles.

(1) The circle of radius R centered at z_0 w/ positive orientation

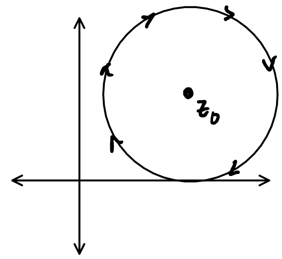
A parameterization is

$$z(t) = z_0 + R e^{it}, \quad t \in [0, 2\pi]$$



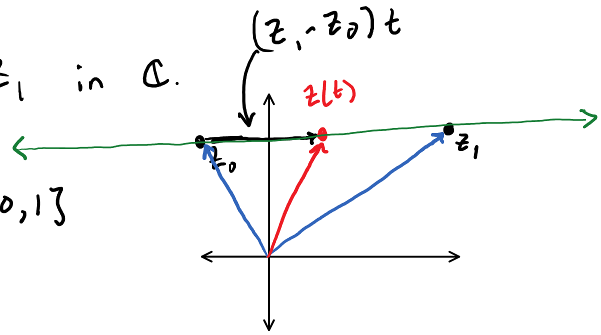
(2) The circle of radius R centered at z_0 w/ negative orientation

$$z(t) = z_0 + Re^{-it}, \quad t \in [0, 2\pi]$$



(3) The line segment from z_0 to z_1 in $\mathbb{C}$.

$$z(t) = z_0 + (z_1 - z_0)t, \quad t \in [0,1]$$



$\boxed{\text{Reparameterization of an arc}}$ Suppose that C is parameterized by

$$z(t) : [a,b] \longrightarrow \mathbb{C}.$$

A map

$$w(s) : [\alpha, \beta] \longrightarrow \mathbb{C}$$

is called an orientation-preserving reparameterization of C if there exists a surjective function

$$\phi : [\alpha, \beta] \longrightarrow [a,b]$$

with continuous derivative such that

$$\underbrace{\phi(\alpha) = a}_{\text{preserves initial point}}, \quad \overbrace{\phi(\beta) = b}^{\text{preserves final pt}}, \quad \underbrace{\phi'(s) > 0}_{\text{strictly increasing}}, \text{ and } \overbrace{w(s) = z(\phi(t))}^{w \text{ and } z \text{ trace out same curve } C}. \quad //$$

$\boxed{\text{Definition (arc length / contours)}}$

(1) If C is parameterized by $z(t) = x(t) + iy(t)$ and $x'(t), y'(t)$ are continuous on $[a,b]$, then C is called a differentiable arc.

(2) The arc length of such a differentiable arc is

$$L = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

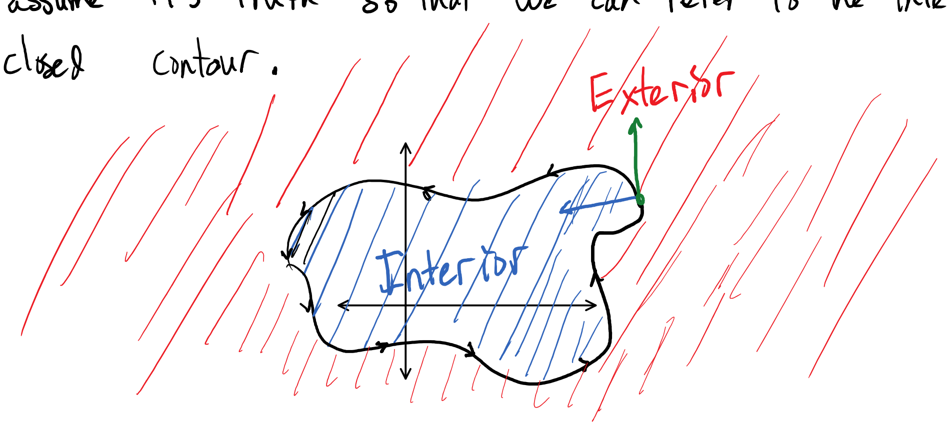
according to the definition from ordinary calculus.

(3) A differentiable arc C parameterized by $z(t)$ is called smooth if $z'(t) \neq 0$ on $[a,b]$.

(4) A contour is an arc consisting of a finite number of smooth arcs joined end to end. A simple closed contour is a contour that does not cross itself except that the initial and final points are the same. //

A deep theorem known as the Jordan Curve Theorem tells us that every simple closed contour C is the boundary of two distinct domains called the interior of C, which is bounded, and the exterior of C, which is unbounded.

The theorem is geometrically evident but the proof is not easy. We will assume it's truth so that we can refer to the interior of a simple closed contour.



Orientation can now be defined via right hand rule: point 4 fingers in the direction of the tangent vector, curl 4 fingers towards interior of curve. If your thumb points up, the orientation is positive. //

Contour Integration

Definition (contour integral) Suppose that $f: U \subseteq \mathbb{C} \to \mathbb{C}$ is a function and C is a contour lying in U. If C is parameterized by $z(t): [a,b] \to \mathbb{C}$ and $f(z(t))$ is piecewise continuous, then the contour integral of f over C is the integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Note: since C is a contour, $z'(t)$ is piecewise continuous so that the integral exists.

Contour integrals are related to ordinary line integrals from calculus. writing $f(z) = u(x,y) + i\,v(x,y)$ and $z(t) = x(t) + i\,y(t)$ we get:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt = \int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt$$

$$= \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\, dt \;\; i \int_a^b u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\, dt$$

$$= \int_a^b u\,dx - v\,dy \; + i \int u\,dy + v\,dx$$

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Proposition (Integral is Independent of parameterization) Suppose that $z: [a,b] \to \mathbb{C}$ parameterizes C and $w: [\alpha,\beta] \to \mathbb{C}$ is an orientation preserving reparameterization of C. Then

$$\int_C f(z)\,dz = \int_C f(w)\,dw.$$

Proof. Choose a function $\phi : [\alpha, \beta] \to [a, b]$ such that

$$\phi(\alpha) = a, \quad \phi(\beta) = b, \quad \phi'(s) > 0, \quad w(s) = z(\phi(s)).$$

Then

$$\int_C f(w)\,dw = \int_\alpha^\beta f(w(t))\,w'(t)\,dt$$

$$= \int_\alpha^\beta f(z(\phi(s)))\,z'(\phi(s)) \cdot \phi'(s)\,ds$$

Set $t = \phi(s)$
$dt = \phi'(s)\,ds$

$\phi(\alpha) = a$
$\phi(\beta) = b$

$$= \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_C f(z)\,dz. \qquad \blacksquare$$

$\boxed{\text{Notation (Contours)}}$

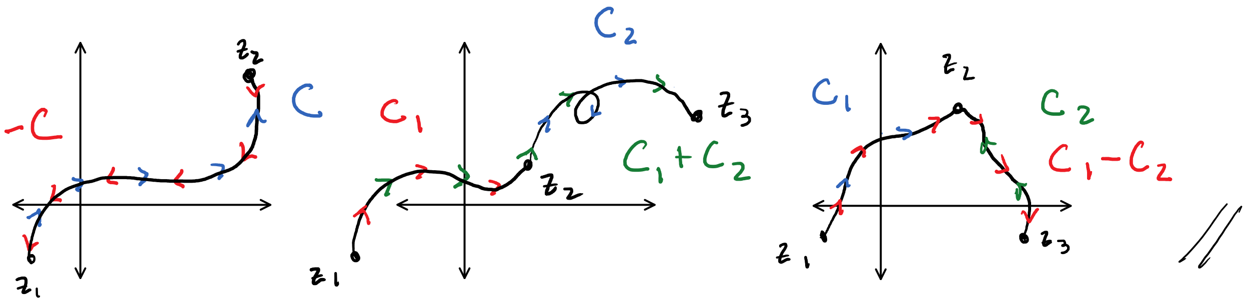
(1) Suppose C is a contour. Then $-C$ denotes the same set of points with opposite orientation. If $z(t) : [a, b] \to \mathbb{C}$ parameterizes C, then $w(t) = z(-t) : [-b, -a] \to \mathbb{C}$ parameterizes $-C$.

(2) If C_1 is a contour from z_1 to z_2 and C_2 from z_2 to z_3, then their sum is

$$C = C_1 + C_2$$

is the contour obtained by traversing C_1 and then C_2. If C_1 and C_2 have the same final point, then the sum of C_1 and $-C_2$ is defined and is written

$$C_1 - C_2 = C_1 + (-C_2).$$



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$\boxed{\text{Proposition (Properties of Contour Integral)}}$ Assume f, g are piecewise continuous on an contour used.

(1) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, $z_0 \in \mathbb{C}$;

(2) $\int_C f(z) + g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$;

(3) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(4) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ if $C = C_1 + C_2$.

Proof.

(1) $\int_C z_0 f(z)\,dz = \int_a^b z_0\, f(z(t))\, z'(t)\,dt$

follows from previous results.

$= z_0 \int_a^b f(z(t))\, z'(t)\,dt$

$= z_0 \int_C f(z)\,dz$

(2) Follows from previous results.

(3) Suppose C is parameterized by $z(t) : [a,b] \to \mathbb{C}$. Then a parameterization for $-C$ is $w(t) = z(-t) : [-b,-a] \to \mathbb{C}$.

$$\int_{-C} f(w)\,dw = \int_{-b}^{-a} f(w(t))\, w'(t)\,dt$$

$s = -t$
$ds = -dt$
$s(-a) = a$
$s(-b) = b$

$$= -\int_{-b}^{-a} f(z(-t))\, z'(-t)\, dt$$

$$= \int_{b}^{a} f(z(s))\, z'(s)\, ds = -\int_{a}^{b} f(z(s))\, z'(s)\, ds$$

from previous results.

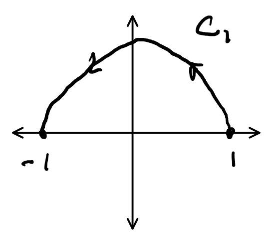
$$= -\int_C f(z)\, dz.$$

(4) Exercise for the motivated student. $\blacksquare$

Examples of Contour Integration

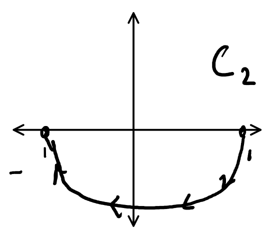
(1) Integrate $f(z) = \frac{1}{z}$ over the following contours:

C_1: upper half of unit circle, from 1 to -1
C_2: lower half of unit circle, from 1 to -1
C_3: $C_1 - C_2$



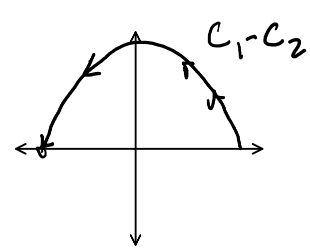
For C_1: parameterize C_1 as $z(t) = e^{it}$, $0 \le t \le \pi$.
Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{it}}\, i e^{it}\, dt = i \int_0^{\pi} 1\, dt = \pi i.$$



For C_2: parameterize C_2 as $z(t) = e^{-it}$, $0 \le t \le \pi$.
Then

$$\int_{C_2} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{-it}}\, (-i e^{-it})\, dt = -i \int_0^{\pi} 1\, dt = -\pi i.$$



For C_3

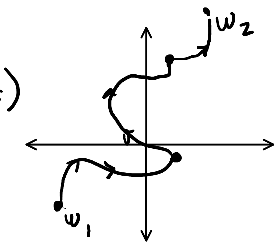
$$\int_{C_3} \frac{1}{z} dz = \int_{C_1 - C_2} \frac{1}{z} dz = \int_{C_1} \frac{1}{z} dz + \int_{-C_2} \frac{1}{z} dz$$

$$= \int_{C_1} \frac{1}{z} dz - \int_{C_2} \frac{1}{z} dz = \pi i - (-\pi i) = 2\pi i.$$

This example shows: the integral may depend on the path taken and not just the endpoints. Also, the integral over a closed contour may be nonzero.

(2) Integrate $f(z) = z$ over <u>any</u> contour C connecting a point w_1 to a point w_2.

First, suppose C is a smooth arc joining w_1 to w_2 and parameterized by $z : [a,b] \to \mathbb{C}$.



Since $\frac{d}{dt}\left(\frac{1}{2} z(t)^2\right) = \frac{1}{2}\left(z(t) z'(t) + z(t) z'(t)\right)$
$= z(t) z'(t)$.

$$\int_C z\,dz = \int_a^b z(t) z'(t)\,dt$$
$$= \frac{1}{2} z(b)^2 - \frac{1}{2} z(a)^2 = \frac{w_2^2 - w_1^2}{2}.$$

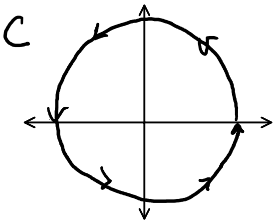
Now, if C is a contour, it can be written a sum of C_i, $i = 1, \dots, n$

where C_i is a smooth arc joining z_i to z_{i+1}, $z_1 = w_1$, $z_{n+1} = w_2$.

Then
$$\int_C z\,dz = \sum_{i=1}^n \int_{C_i} z\,dz = \sum_{i=1}^n \frac{z_{i+1}^2 - z_i^2}{2}$$
$$= \frac{z_{n+1}^2 - z_1^2}{2}$$
$$= \frac{w_2^2 - w_1^2}{2}.$$

This example shows that some integrals depend only on the endpoints and not the path taken. Also, if $w_2 = w_1$, then we showed that
$$\int_C z\,dz = 0$$
for any closed contour C. //

(3) Integrate $f(z) = z^m \bar{z}^n$, $m, n \in \mathbb{Z}$, over the unit circle.



Parameterize C as $z(t) = e^{it}$, $0 \le t \le 2\pi$.

Then

$$\int_C z^m \bar{z}^n dz = \int_0^{2\pi} (e^{it})^m (\overline{e^{it}})^n \, i e^{it} dt$$

$$= i \int_0^{2\pi} e^{imt} (e^{\overline{it}})^n e^{it} dt$$

$$= i \int_0^{2\pi} e^{imt} \, e^{-int} \, e^{it} dt$$

$$= i \int_0^{2\pi} e^{i(m-n+1)t} dt$$

Case 1: $m = n-1$

$$= i \int_0^{2\pi} 1 \, dt = 2\pi i$$

Case 2: $m \neq n-1$

$$= i \left(\left[\frac{1}{i(m-n+1)} e^{i(m-n+1)t} \right]_0^{2\pi} \right)$$

$$= \frac{1}{m-n+1} \left(e^{i(m-n+1) \cdot 2\pi} - e^0 \right)$$

integer mult. of 2π

$$= \frac{1}{m-n+1} (1 - 1) = 0.$$

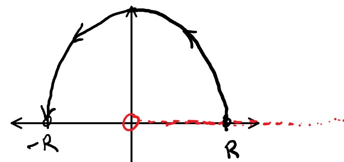
Some examples involving a branch of a multiple valued function:

(4) Integrate the branch of square root

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} \qquad (|z|>0,\ 0<\arg z<2\pi)$$

along the contour:

$$C: \quad z(t) = Re^{it} \qquad (R>0,\ \underbrace{0 \le t \le \pi})$$



The problem is that the integrand $\underline{f(z(t))z'(t)}$ is not defined when $t=0$. But the function is piecewise continuous on $[0,\pi]$:

$$f(z(t))\,z'(t) = e^{\frac{1}{2}\log Re^{it}}\, Rie^{it} = e^{\frac{1}{2}(\ln R + it)}\, Re^{it}$$
$$= \sqrt{R}\ e^{\frac{1}{2}it}\, Re^{it}$$
$$= R^{3/2}\ e^{3/2\, it} = R^{3/2}\left(\cos\tfrac{3}{2}t + i\sin\tfrac{3}{2}t\right)$$

The real/im parts of the function are continuous on $(0,\pi]$ and the limits approaching 0 from the right are as expected. So So the integrand is piecewise cont. on $[0,\pi]$ and the integral exists. We have

$$\int_C f(z)\,dz = R^{3/2}\int_0^{\pi} e^{3/2\, it}\,dt = R^{3/2}\left[\frac{2}{3i}e^{3/2\, it}\right]_0^{\pi}$$
$$= R^{3/2}\,\frac{2}{3i}\left(e^{3/2\,\pi i} - 1\right) = R^{3/2}\,\frac{2}{3i}\,(-i-1). \quad //$$

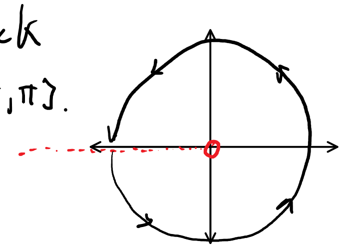
(5) Integrate the principal branch of z^{i-1}:

$$f(z) = z^{i-1} = e^{(i-1)\,\mathrm{Log}\, z}$$

along the contour

$$C: \quad z(t) = e^{it}, \qquad -\pi \le t \le \pi.$$

The curve crosses the branch cut. We need to check if $f(z(t)) \cdot z'(t)$ piecewise continuous on $[-\pi, \pi]$.



We have

$$f(z(t))\, z'(t) = e^{(i-1)\,\mathrm{Log}\, e^{it}}\; i e^{it}$$

$$= e^{(i-1)(\ln 1 + it)}\; i e^{it} = e^{(i^2 - i)t} \cdot i e^{it} = i e^{-t}.$$

Hence,

$$\int_C f(z)\, dz = i \int_{-\pi}^{\pi} e^{-t} = i\left[-e^{-t}\right]_{-\pi}^{\pi}$$

$$= i\left[-e^{-\pi} + e^{\pi}\right]$$

$$= 2i \sinh \pi.$$

//

Estimating Contour Integrals

Lemma (Triangle Ineq. for Integrals) Suppose $w : [a,b] \to \mathbb{C}$ is piecewise continuous. Then

$$\left| \int_a^b w(t)\, dt \right| \leq \int_a^b |w(t)|\, dt.$$

Proof. First, assume $\int_a^b w(t)\, dt = 0$. Then the lemma holds since $|w(t)| \geq 0$ for all $t \in [a,b]$ and so its integral is also nonnegative. Otherwise, $\int_a^b w(t)\, dt \neq 0$ so we can use polar coordinates:

$$r_0 e^{i t_0} = \int_a^b w(t)\, dt.$$

Then

$$\left|\int_a^b w(t)\,dt\right| = \left|r_0 e^{it_0}\right|$$

$$= r_0$$

$$= \operatorname{Re} r_0$$

$$= \operatorname{Re}\left(r_0 e^{it_0} \cdot e^{-it_0}\right)$$

$$= \operatorname{Re}\left(\int_a^b e^{-it_0} w(t)\,dt\right)$$

$$= \int_a^b \operatorname{Re}\left(e^{-it_0} w(t)\right) dt$$

Week 1

$$\leq \int_a^b \left|e^{-it_0}\right| |w(t)|\,dt$$

$$= \int_a^b |w(t)|\,dt.$$

∎

$\boxed{\text{Theorem (Triangle Ineq. for Contour Integrals)}}$ Suppose that C is a contour of length L and f is piecewise continuous on C. Then

finite?

$$\left|\int_C f(z)\,dz\right| \leq \max_{z \in C} |f(z)| \cdot L.$$

Proof. Suppose $z : [a,b] \to \mathbb{C}$ parameterizes C. By assumption $f(z(t))$ is piecewise continuous on $[a,b]$. Hence,

$$\max_{z \in C} |f(z)| = \max_{t \in [a,b]} |f(z(t))| \quad \text{is finite}$$

because $f(z(t))$ is piecewise cont. on a closed interval. Hence,

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$

Lemma

$$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq \max_{z \in C} |f(z)|} |z'(t)|\,dt$$

$$\leq \max_{z \in C} |f(z)| \cdot \int_a^b |z'(t)|\,dt$$

$$= \max_{z \in C} |f(z)| \cdot L.$$

∎

$\boxed{\text{Example}}$

(1) Finding an upper bound for

$$\int_C \frac{z^2+1}{z^3+2}\,dz$$

Semicircle radius 2

where C is the semicircle $z(t) = 2e^{it}$, $0 \le t \le \pi$.

All we need to do is find $M \geq 0$ such that

$$\left| \frac{z^2+1}{z^3+2} \right| \le M \quad \text{for all } z \in C.$$

Suppose $z \in C$ so that $|z| = 2$. Then

$$|z^2+1| \le |z|^2 + 1 = 5.$$

Also,

$$|z^3+2| \ge \big| |z|^3 - 2 \big| = |2^3 - 2| = 6.$$

Together,

$$\left| \frac{z^2+1}{z^3+2} \right| \le \frac{5}{6} \quad \text{for all } z \in C.$$

Hence,

$$\left| \int_C \frac{z^2+1}{z^3+2}\,dz \right| \le \frac{5}{6} \cdot \underbrace{2\pi}_{L} \quad \text{by the Theorem.} \quad //$$

(2) Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\,dz = 0$$

where C_R is the circle $z(t) = Re^{it}$, $0 \le t \le 2\pi$.

Note: the length of C_R is $2\pi R$. Let $z \in C_R$ so that $|z| = R$. Then

$$|z^2+z| \le |z|^2 + |z| = R^2 + R$$

and

$$|z^4+2z^2+1| = |(z^2+1)(z^2+1)|$$
$$= |z^2+1|^2 \qquad (\text{Assume } R>2)$$
$$\geq \big||z|^2-1\big|^2 = |R^2-1|^2$$
$$= (R^2-1)^2.$$

then $$\left|\int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\right| \leq 2\pi R\cdot\left(\frac{R^2+R}{(R^2-1)^2}\right) \xrightarrow{R\to\infty} 0. \quad //$$

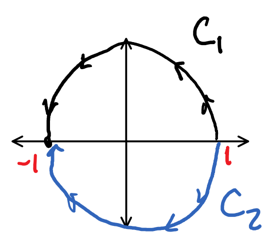
Antiderivatives & Fundamental Theorem of Contour Integrals

Suppose C is a contour joining z_1 to z_2. In general, the value of the integral

$$\int_C f(z)\,dz$$

depends on C. For example, we have seen that

$$\int_{C_1} \frac{1}{z}\,dz = \pi i$$



while

$$\int_{C_2} \frac{1}{z}\,dz = -\pi i.$$

But we have also seen that

$$\int_C z\,dz = \frac{z_2^2 - z_1^2}{2}$$

difference between thes functions turns out to be that $f(z)=z$ has an antiderivative on $\mathbb{C}$, while $g(z)=\frac{1}{z}$ does not have an antiderivative on any domain containing C_1 and C_2. //

Definition (Antiderivative) Suppose that f is a continuous function on a domain D. An analytic function $F : D \to \mathbb{C}$ is called an *antiderivative* of f if $F'(z) = f(z)$ for all $z \in D$. //

Definition (Independence of Path) Let $f : D \to \mathbb{C}$ be a continuous function on a domain D and fix $z_1, z_2 \in D$. If

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

whenever C_1 and C_2 are contours in D joining z_1 to z_2, then the integrals of f from z_1 to z_2 are *independent of path* and we denote the unique value by

$$\int_{z_1}^{z_2} f(z)\, dz .$$

//

So, for instance we would write

$$\int_{z_1}^{z_2} z\, dz = \frac{z_2^{\,2} - z^{2}}{2} .$$

Since we have already proved the integrals of z from z_1 to z_2 are independent of path.

Theorem (Fundamental Theorem of Contour Integrals)
Suppose f is continuous on a domain D. The following are equivalent:

(1) f has an antiderivative $F : D \to \mathbb{C}$.

(2) For all $z_1, z_2 \in D$, the integrals of f from z_1 to z_2 are independent of path and the unique value is given by

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

(3) If C is any closed contour lying in D, then

$$\int_C f(z)\,dz = 0.$$

Proof. ((1) ⇒ (2)) Suppose f has an antiderivative $F: D \to \mathbb{C}$. Let $z_1, z_2 \in D$ and let C be any contour joining z_1 to z_2 and lying in D.

First assume C is a smooth arc parameterized by $z: [a,b] \to \mathbb{C}$.

PSet 4 P2

Then $$\frac{d}{dt}\left(F(z(t))\right) = F'(z(t))\,z'(t) = \underbrace{f(z(t))\,z'(t)}_{\text{the integrand}}$$

Hence (*) $$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = F(z(b)) - F(z(a))$$
$$= F(z_2) - F(z_1).$$

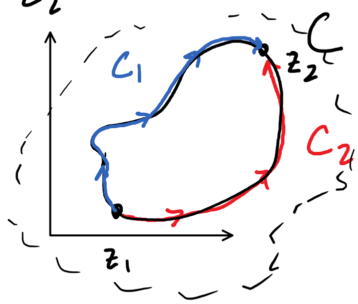
Now, assume C is a contour. Then $C = C_1 + \cdots + C_n$ where C_i is a smooth arc joining w_i and w_{i+1}. Then

where $w_{n+1} = z_2$ and $w_1 = z_1$

$$\int_C f(z)\,dz = \int_{\sum_{i=1}^n C_i} f(z)\,dz = \sum_{i=1}^n \int_{C_i} f(z)\,dz$$
$$\overset{(*)}{=} \sum_{i=1}^n F(w_{i+1}) - F(w_i)$$
$$= F(w_{n+1}) - F(w_1)$$
$$= F(z_2) - F(z_1).$$

Since $F(z_2) - F(z_1)$ depends only on z_1 and z_2, we have proved the claim.

((2) $\Rightarrow$ (3)) Assume (2) and let C be any closed contour lying in the domain. Choose any 2 distinct pts z_1 and z_2 on C. Let C_1 and C_2 be contours from z_1 to z_2 such that $C = C_1 - C_2$. Then



$$\int_C f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

by assumption

$$= \int_{z_1}^{z_2} f(z)\,dz - \int_{z_1}^{z_2} f(z)\,dz = 0.$$

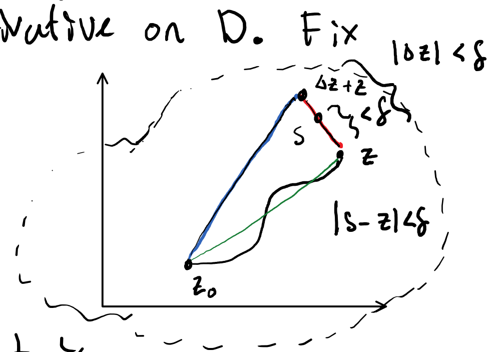
((3) $\Rightarrow$ (2)) Assume (3) and let $z_1, z_2 \in D$. Suppose C_1 and C_2 are two contours in D joining z_1 to z_2. Then $C = C_1 - C_2$ is a closed contour. By assumption

by assumption

$$0 = \int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz.$$

So $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ as claimed.

((2) $\Rightarrow$ (1)) Assume (2) (and (3) since they are equivalent) - I need to show is that f has an antiderivative on D. Fix any point $z_0 \in D$ and define



$$F(z) = \int_{z_0}^{z} f(s)\,ds.$$

By (2), this function is well-defined. We need to show that $F'(z) = f(z)$, that is

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

Let $\varepsilon > 0$ and $z \in D$. Since f is continuous at z, so $\delta > 0$ such that

$$|s - z| < \delta \implies |f(s) - f(z)| < \varepsilon.$$

To compute the difference quotient, let Δz be a complex number close enough to z so that $z + \Delta z \in D$. Then

(both integrals taken over straight line paths)

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

integral over a closed path is 0

$$= \int_{z}^{z+\Delta z} f(s)\,ds.$$

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Next,

$$f(z) = \frac{f(z)\,\Delta z}{\Delta z} = \frac{1}{\Delta z} f(z) \int_z^{z+\Delta z} 1\,ds$$

$$= \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\,ds.$$

Now, assume Δz is is so close to z that $|\Delta z| < \delta$. It follows that $|s - z| < \delta$ for any point s on the line segment between z and $\Delta z + z$ (see picture). Hence, by continuity, $|f(s) - f(z)| < \varepsilon$. Using the preceding computations and the Triangle Ineq. for contour integrals, we obtain:

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{\left| \int_z^{z+\Delta z} f(s)\,ds - \int_z^{z+\Delta z} f(z)\,ds \right|}{|\Delta z|}$$

$$= \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} f(s) - f(z)\,ds \right|$$

length of line segment from z to $z+\Delta z$

$$\overset{\text{T.I.}}{\leq} \frac{1}{|\Delta z|}\, \varepsilon \cdot |\Delta z|$$

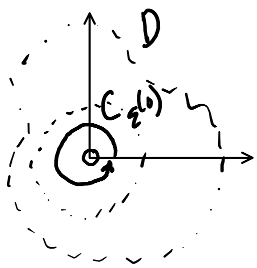
$$= \varepsilon.$$

We have shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Delta z| < \delta$ implies $\left| \dfrac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon$.

That is $F'(z) = f(z)$ for all $z \in D$. ∎

Example

(1) The function $f(z) = \frac{1}{z}$ has no antiderivative on $\mathbb{C}\setminus\{0\}$. In fact, it has no antiderivative on any domain containing a deleted neighborhood of 0. Take a circle $C_\varepsilon(0)$ so small that it lies in this domain. Then



$$\int_{C_\varepsilon} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, \varepsilon i e^{it}\, dt$$

$$= i\int_0^{2\pi} 1\, dt = 2\pi i.$$

By FToCI, $f(z)$ has no antiderivative on such a domain. The problem is as follows: any branch $F(z) = \log z$ has derivative

$$F'(z) = \frac{1}{z}.$$

But $F(z)$ is not even defined on it branch cut. In fact, it is not possible to extend such a branch of log to an analytic function on all of $\mathbb{C}\setminus\{0\}$.

(2) $f(z) = \cos z$ is continuous on $\mathbb{C}$ and also $\sin z$ is entire. Moreover,

$$\frac{d}{dz}\sin z = \cos z = f(z).$$

So f has an antiderivative on $\mathbb{C}$. So for instance

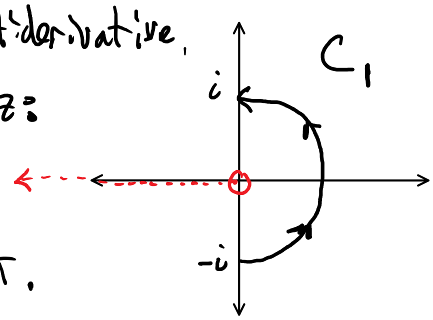
$$\int_0^{\pi i} \cos z \, dz = \sin \pi i - \sin 0$$
$$= \sin \pi i.$$

//

(3) Although $f(z) = \frac{1}{z}$ has no antiderivative on any domain containing 0, we can integrate f over a circle by using two different antiderivatives.

Let C_1 be parameterized by $z(t) = e^{it}$, $t \in [-\pi/2, \pi/2]$. On $\mathbb{C} \setminus (-\infty, 0]$, $f(z)$ has an antiderivative, namely the principal branch of $\log z$:

$$\text{Log } z = \ln r + i\theta, \quad r > 0, \quad -\pi < \theta < \pi.$$

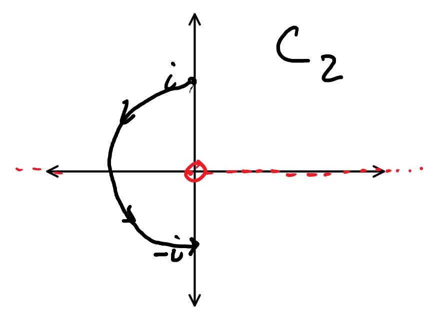


By the fundamental theorem,

$$\int_{C_1} \frac{1}{z} \, dz = \text{Log } i - \text{Log } -i$$
$$= \ln|i| + i\pi/2 - (\ln|-i| + i(-\pi/2))$$
$$= \pi i.$$

On the domain $\mathbb{C} \setminus [0, \infty)$, $f(z)$ has an antiderivative, namely

$$\log z = \ln r + i\theta, \quad r > 0, \quad 0 < \theta < 2\pi.$$



Hence,
$$\int_{C_2} \frac{1}{z} \, dz = \log -i - \log i$$

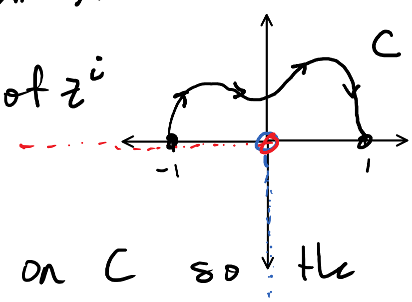
$$= i\,3\pi/2 - i(\pi/2) = \pi i$$

Hence, $\displaystyle\int_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} dz$

$$= \pi i + \pi i = 2\pi i.$$

(4) Sometimes we can integrate a branch of a multiple valued function f even when the contour crosses the branch cut.

Let C be any curve joining -1 to 1 and lying above the x-axis (except the endpoints).



We will integrate the principal branch of z^i

$$f(z) = z^i = e^{i\,\mathrm{Log}\,z}$$

The function is piecewise continuous on C so the integral exists. The function $f(z)$ has an antiderivative on its domain of definition, but C does not live in that domain. But, the branch

$$g(z) = z^i = e^{i\log z}, \quad |z|>0 \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

The functions $f(z)$ and $g(z)$ agree everywhere on C (except at $z=-1$). But $g(z)$ has an antiderivative on a domain containing C, so

$$\int_C f(z)\,dz = \int_C g(z)\,dz$$

$$= \frac{1}{i+1}(1)^{i+1} - \frac{1}{i+1}(-1)^{i+1}$$

$$= \frac{1}{i+1}\left(1 - e^{-\pi}\right)$$

Cauchy-Goursat Theorem

The Cauchy-Goursat integral theorem gives a sufficient condition for the integral of a function over a simple closed curve to be zero. The theorem has powerful implications. Ultimately it leads to

- The Cauchy Integral formula.
- The theory of residues for computing contour integrals.
- A method to evaluate real-valued functions of a real variable, using contour integration.

Historically, the theorem was first proved by Cauchy with a weaker hypothesis. We prove this first.

formal symbol

Recall: (1) Contour integrals are related to line integrals. You can remember this by writing $f = u + iv$ and $dz = dx + idy$. Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx.$$

(2) Greens Theorem: suppose C is a simple closed contour in $\mathbb{R}^2$ and let R be the region enclosed by C. If P and Q have continuous first order partial derivatives on R. Then

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$$

$\boxed{\text{Theorem (Weak Cauchy Integral Theorem)}}$ Let C be a simple closed contour and let R denote the closed region consisting of C and its interior. If f is analytic on R and f' is continuous on R, then

$$\int_C f(z)\,dz = 0.$$

Proof. If f is analytic on R, then $u_x = v_y$ and $u_y = -v_x$ on R and

$$f' = u_x + i v_x$$
$$= v_y - i u_y.$$

Since f' is continuous, so are u_x, v_x, v_y, and u_y. Hence, by the above

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx.$$

(Green's Thm)

$$= \iint_R -v_x - u_y\, dA + i \int u_x - v_y\, dA$$

(Cauchy Riemann)

$$= \iint_R -V_x + V_x \, dA + i \iint_R U_x - U_x \, dA$$

$$= 0.$$

■

Goursat was the first to prove that the continuity of f' can be omitted. This turns out to be essential for the theory of analytic functions. The problem is that it may be difficult to prove that the derivative of an analytic function is continuous!

☆

$\boxed{\text{Theorem (Cauchy-Goursat Theorem)}}$ Let C be a simple closed contour and R the closed region consisting of C and its interior. If f is analytic on R, then

$$\int_C f(z)\, dz = 0.$$

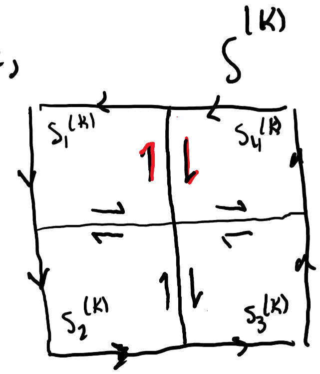
Proof. (For simplicity, we will assume C is a square.)

The idea is to "divide and conquer". The idea is to break the curve into a finite number of smaller squares on which we can estimate the integral. We first construct a sequence of positively oriented curves $S^{(k)}$, each of which is the boundary of a square region $R^{(k)}$.

To begin, set $S^{(0)} \overset{def}{=} C$. Then, inductively, after the first k squares have been chosen, define the $(k+1)^{th}$ square as follows:

Divide $S^{(K)}$ into four congruent squares w/ positive orientation: $S_1^{(K)}, S_2^{(K)}, S_3^{(K)}, S_4^{(K)}$.

Notice that the integrals of f over the shared boundaries of these squares cancel. Hence,



$$\int_{S^{(K)}} f(z)\,dz = \sum_{i=1}^{4} \int_{S_i^{(K)}} f(z)\,dz .$$

Define $$S^{(K+1)} \overset{\text{def}}{=} \max_{i=1}^{4} \left| \int_{S_i^{(K)}} f(z)\,dz \right| .$$

At this point, the sequence $S^{(0)}, \dots, S^{(K)}, \dots$ has been defined.

Notice that
$$\left| \int_{S^{(K)}} f(z)\,dz \right| \leq \sum_{i=1}^{4} \left| \int_{S_i^{(K)}} f(z)\,dz \right|$$
$$\leq 4 \left| \int_{S^{(K+1)}} f(z)\,dz \right| .$$

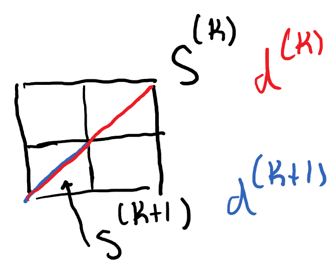
So by induction, for any $n \in \mathbb{N}$.

$$\boxed{(*) \quad \left| \int_{S^{(0)}=C} f(z)\,dz \right| \leq 4^n \left| \int_{S^{(n)}} f(z)\,dz \right| .}$$

Next, we record some more facts. Denote by $d^{(n)}$ the length of the diagonal of the nth square $S^{(n)}$ and denote by $p^{(n)}$ its perimeter.

Then
$$d^{(n)} = \frac{1}{2^n} d^{(0)}$$
$$p^{(n)} = \frac{1}{2^n} p^{(0)} .$$

Also, $d^{(n)} \longrightarrow 0$ as $n \longrightarrow \infty$.



Next, consider the associated sequence

$$R^{(0)} \supset R^{(1)} \supset \cdots \supset R^{(K)} \supset \cdots\cdots .$$

Each $R^{(K)}$ is compact (closed and bounded) and hence there is a unique point

$$z_0 \in \bigcap_{i=0}^{\infty} R^{(i)} .$$

A Fact assumed from analysis.

Since $z_0 \in R = R^{(0)}$, f is analytic at z_0. So we can define the following function:

$$\psi(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0) & , z \neq z_0 \\ 0 & , \text{otherwise.} \end{cases}$$

Then $\lim\limits_{z \to z_0} \psi(z) = f'(z_0) - f'(z_0) = 0 = \psi(z_0)$ so ψ is continuous at z_0. We can write

$$f(z) = f(z_0) + \psi(z)(z-z_0) + f'(z_0)(z-z_0).$$

Note that $f(z_0)$, $f'(z_0)(z-z_0)$ have antiderivatives on $\mathbb{C}$. Hence, by the fundament theorem

$$\int_{S^{(n)}} f(z)\,dz = \underbrace{\cancel{\int_{S^{(n)}} f(z_0)\,dz}}_{0} + \int_{S^{(n)}} \psi(z)(z-z_0)\,dz + \underbrace{\cancel{\int_{S^{(n)}} f'(z_0)(z-z_0)\,dz}}_{0}$$

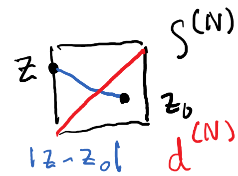
$$= \int_{S^{(n)}} \psi(z)(z-z_0)\,dz .$$

Let $\varepsilon > 0$. Since ψ is continuous at z_0, choose $\delta > 0$ such that

$$|z - z_0| < \delta \implies |\psi(z)| < \varepsilon.$$

Since $d^{(n)} \to 0$, choose $N \in \mathbb{N}$ such that $n \geq N$,

$|d^{(n)}| < \delta$. Hence, if $z \in S^{(N)}$, then $|z - z_0| < |d^{(N)}| < \delta$, and hence $|\psi(z)| < \varepsilon$.



Hence, we obtain

$$\left| \int_{S^{(N)}} f(z)\,dz \right| = \left| \int_{S^{(N)}} \psi(z)(z - z_0)\,dz \right|$$

$$\underset{\text{Triangle ineq}}{\leq} \max_{z \in S^{(N)}} |\psi(z)||z - z_0| \cdot \text{length } S^{(N)}.$$

$$\leq \varepsilon \, d^{(N)} p^{(N)}$$

$$= \varepsilon \cdot \frac{1}{2^N} \cdot \frac{1}{2^N} d^{(0)} p^{(0)}.$$

Hence, by (*)

$$\left| \int_{S^{(0)} = C} f(z)\,dz \right| \leq 4^N \left| \int_{S^{(N)}} f(z)\,dz \right|$$

$$\leq \varepsilon \cdot 4^n \frac{1}{2^n} \frac{1}{2^n} d^{(0)} p^{(0)} = \varepsilon d^{(0)} p^{(0)}.$$

The right hand side depends only on ε. Take $\varepsilon \to 0$ to obtain

$$\left| \int_C f(z)\,dz \right| = 0 \implies \boxed{\int_C f(z)\,dz.}$$

Simply Connected Domains

Definition (simply connected domain) A domain D is called simply connected if it has the following property: if C is any simple closed contour lying in D and z is interior to C, then $z \in D$. //

Intuitively, a simply connected domain is a domain that has no "holes."

Examples (simply connected)
- Open disks
- Complex plane
- interior of any simple closed curve.

(not simply connected domains)
- deleted open disks
- $\mathbb{C} \setminus \{p\}$

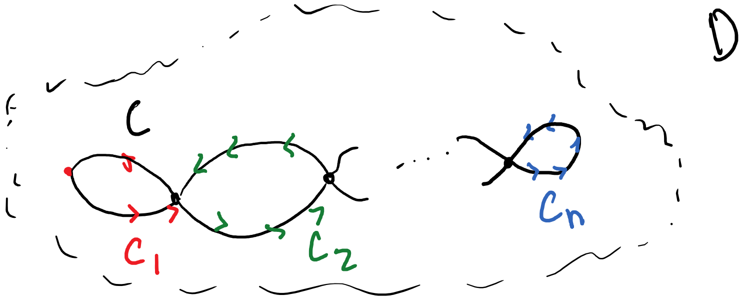
A result similar to the Cauchy-Goursat theorem can be proved for closed contours that may not be simple, provided that they lie in a simply connected domain.

Theorem (Cauchy-Goursat for Simply Connected Domains)
Suppose that f is analytic on a simply connected domain D. If C is any closed contour lying in D, then

$$\int_C f(z)\,dz = 0.$$

Proof. (We will assume this holds for contours with infinitely many self-intersections, without proof)

Assume C is a closed contour w/ finitely many self intersections.



Then C is made up of a finite number of simple closed contours $C_1, \dots, C_n$. So

$$\int_C f = \sum_{i=1}^{n} \int_{C_i} f = 0$$

by applying Cauchy-Goursat to each integral $\int_{C_i} f$. ∎

$\boxed{\text{Corollary (Antiderivatives of analytic functions)}}$ If f is analytic on a simply connected domain D, then f has an antiderivative on D.

Proof. By the preceding theorem, $\int_C f(z)dz = 0$ for any closed contour lying in D. By the Fundamental Thm of Contour Integrals, this is equivalent to f having an antiderivative on D. ∎

$\boxed{\text{Corollary (Entire functions have antiderivatives)}}$ Suppose that f is entire. Then f has an antiderivative on $\mathbb{C}$.

Proof. The complex plane $\mathbb{C}$ is simply connected.

Apply the preceding corollary. ∎

Multiply Connected Domains

Definition (Multiply Connected) A domain D is called *multiply connected* if it is not simply connected.

The Cauchy-Goursat theorem is easily generalized to multiply connected domains with a finite number of holes.

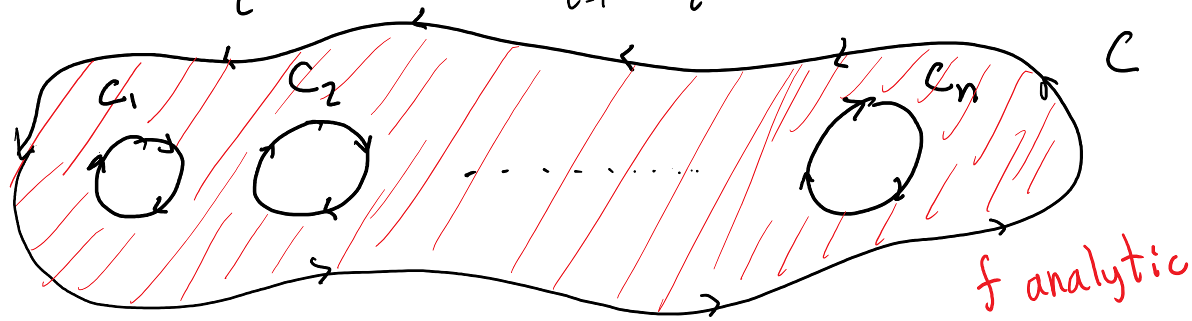
Theorem (Generalized Cauchy-Goursat) Suppose that

(1) C is a simple closed positively oriented contour.

(2) $C_1, \dots, C_n$ are simple closed negatively oriented contours enclosing regions $R_1, \dots, R_n$. Furthur assume that the regions are pairwise disjoint and interior to C.

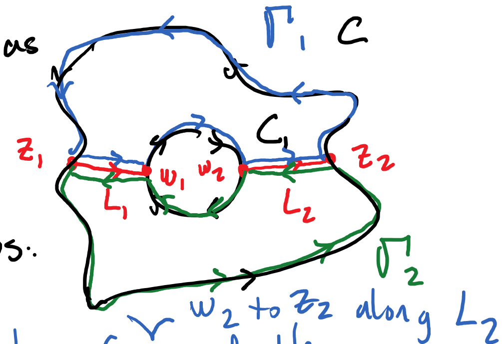
If f is analytic on each contour and the region consisting of all points interior to C but exterior to each C_i, then

$$\int_C f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0.$$



Proof. By induction.

Base case: $n=1$. Assume C and C_1 are contours satisfying the hypotheses. Let z_1, z_2, w_1, w_2 be as in the picture. Join z_1 to w_1 with a polygonal line L_1. Also, join w_2 to z_1 with a polygonal line L_2. Define contours as follows:



Γ_1: follow z_1 to w_1 along L_1, then w_1 to w_2 along C_1 and then z_2 to z_1 along C.

Γ_2: follow z_2 to w_2 along $-L_2$ and w_2 to w_1 along C_1 and w_1 to z_1 along $-L_1$ and z_1 to z_2 along C.

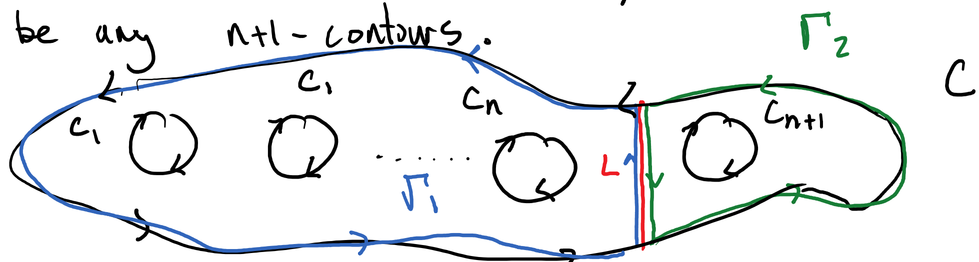
Then f is analytic inside and on the simple closed curves Γ_1 and Γ_2, so by the Cauchy-Goursat theorem,

$$\int_C f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz$$

$$\overset{\text{Cauchy-Goursat}}{=} 0 + 0$$

$$= 0.$$

Inductive step: assume $n \geq 1$ and $\int_C f + \sum \int_{C_i} f = 0$ for any n curves $C_1, \ldots, C_n$ satisfying the hypotheses. Let $C_1, \ldots, C_n, C_{n+1}$ be any $n+1$-contours.



Introduce a polygonal line L that seperates $C_1, \ldots, C_n$ from C_{n+1}. Let Γ_1 and Γ_2 be curves defined as in the picture. Then

$$\int_C f = \int_{\Gamma_1} f + \int_{\Gamma_2} f.$$

By inductive hypothesis $\int_{\Gamma_1} f = -\sum_{i=1}^{n} \int_{C_i} f$. By the case $n=1$, $\int_{\Gamma_2} f = -\int_{C_{n+1}} f$. Hence,

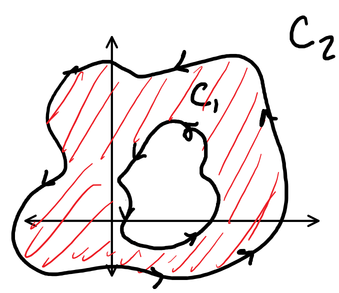
$$\int_C f = -\sum_{i=1}^{n} \int_{C_i} f + \left(-\int_{C_{n+1}} f\right) = -\sum_{i=1}^{n+1} \int_{C_i} f .$$

∎

Corollary (Principle of Deformation of Paths)

Suppose that C_1 and C_2 are positively oriented simple closed contours with C_1 interior to C_2. If f is analytic on the region consisting of C_1, C_2 and all the points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz .$$



Proof. Apply the Generalized Cauchy-Goursat theorem to C_2 and $-C_1$ to get

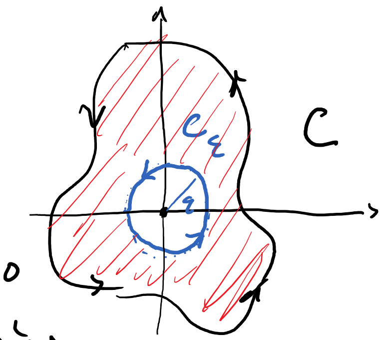
$$\int_{C_2} f + \int_{-C_1} f = 0 .$$

Since $\int_{-C_1} f = -\int_{C_1} f$, this proves the claim.

∎

Among other things, the principle of deformation of paths is useful for integrating over complicated contours. Often, we can just replace the contour with a circle.

$\boxed{\text{Example}}$ Let C be any simple closed contour whose interior contains 0. We show that



$$\int_C \frac{1}{z}\,dz = 2\pi i.$$

Since 0 is interior to C, we can choose $\varepsilon > 0$ so small that $C_\varepsilon(0)$ is contained in the interior of C. The region consisting of C, C_ε and the points in between doesn't contain 0, so $\frac{1}{z}$ is analytic there. By deformation of paths

$$\int_C \frac{1}{z}\,dz = \int_{C_\varepsilon} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\,dt$$

$$= \int_0^{2\pi} i\,dt = 2\pi i.$$

//

More generally, the generalized Cauchy-Goursat theorem and its corollary provide a technique for integrating functions over contours whose interior contains singularities of that function. The idea is to introduce small circles around the singular points, and apply the theorem. It is usually easy to integrate over a circle.

We will use this technique to prove the Cauchy Integral formula and the residue theorem. Exciting!

//

Cauchy's Integral Formula

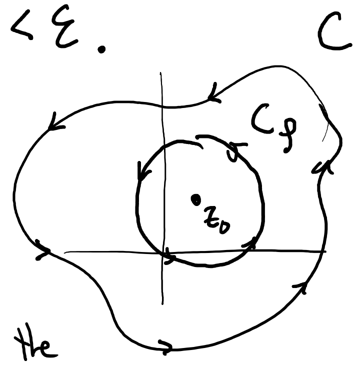
Let C be a simple closed contour. The Cauchy Integral formula is a remarkable theorem. It asserts that, if a function is analytic inside and on C, then its values interior to C are totally determined by its values on C.

Theorem (Cauchy Integral Formula) Let C be a simple closed positively oriented contour. If f is analytic at all points on and interior to C and $z_0 \in \operatorname{Int} C$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz .$$

Proof. The idea is to show that, for all $\varepsilon > 0$

$$\left| \int_C \frac{f(z)}{z - z_0}\, dz - f(z_0) 2\pi i \right| < \varepsilon .$$



Let $\varepsilon > 0$. Let $\rho > 0$ so small that $C_\rho(z_0)$ (the circle of radius ρ centered at z_0) is interior to the curve C. Assume C_ρ has positive orientation. Note that $\frac{f(z)}{z - z_0}$ is analytic on the region consisting of C, C_ρ, and the points in between, so by deformation of paths

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz .$$

Since f is continuous at z_0, choose $\delta > 0$ such that

$$|z - z_0| < \delta \quad \text{implies} \quad |f(z) - f(z_0)| < \frac{\varepsilon}{2\pi} .$$

Shrink ρ so that $\rho < \overline{\rho}$. Then every point $z \in C_\rho(z_0)$ satisfies $|f(z) - f(z_0)| < \frac{\varepsilon}{2\pi}$. Then

$$\left| \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0)2\pi i \right| = \left| \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - f(z_0)2\pi i \right|$$

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$$= \left| \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - f(z_0) \int_{C_\rho} \frac{1}{z-z_0}\,dz \right|$$

$$= \left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right|$$

Triangle Ineq

$$\leq \max_{z \in C_\rho} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| \cdot 2\pi\rho$$

$$< \frac{\varepsilon}{2\pi\rho} \cdot 2\pi\rho = \varepsilon.$$

∎

Among other things, the Cauchy Integral formula is useful for computing integrals.

$\boxed{\text{Example}}$

(1) $\displaystyle\int_C \frac{\cos z}{z(z^2+2)}\,dz$, C: the positively oriented unit circle

Consider $f(z) = \frac{\cos z}{z^2+2}$. Then f is analytic on and interior to C. By Cauchy integral formula

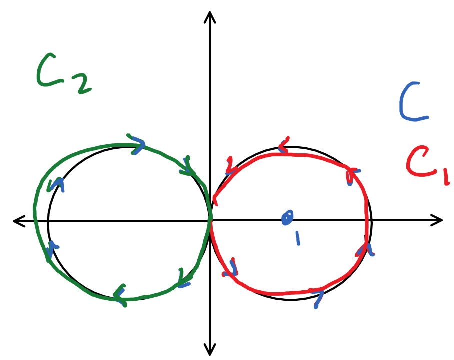
$$\int_C \frac{\cos z}{z(z^2+2)}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i \cdot f(0) = \pi i.$$

(2) $\displaystyle\int_C \frac{e^{z^2}}{z-1}\,dz$, C: the positively oriented circle $|z|=2$.

Consider $f(z)=e^{z^2}$. Then $f(z)$ is entire and hence analytic on and interior to C. Also, 1 is interior to C so

$$\int_C \frac{e^{z^2}}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\, e.$$

(3) $\displaystyle\int_C \frac{z^2+1}{z^2-1}\,dz$, C:



The contour C is not simple, but it can be decomposed as a sum of simple closed contours:

$$C = C_1 + C_2.$$

So

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_{C_1} \frac{z^2+1}{z^2-1}\,dz + \int_{C_2} \frac{z^2+1}{z^2-1}\,dz.$$

Consider C_1: consider $f(z)=\dfrac{z^2+1}{z+1}$. Then $f(z)$ is analytic inside and on C_1. Also 1 is interior to C_1. By Cauchy's formula,

$$\int_{C_1} \frac{z^2+1}{z^2-1}\,dz = \int_{C_1} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i.$$

For C_2: consider $g(z)=\dfrac{z^2+1}{z-1}$. Then g is analytic inside and on C_2. Also, -1 is interior to C_2. Hence,

$$\int_{C_2} \frac{z^2+1}{z^2-1}\,dz = -\int_{-C_2} \frac{g(z)}{z-(-1)}\,dz = -2\pi i\, g(-1) = 2\pi i.$$

Hence $\int_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i + 2\pi i = 4\pi i$. //

Theorem (Generalized Cauchy Integral Theorem)

Suppose that f is analytic interior to and on a simple closed positively oriented contour C. If z_0 is interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Theorem (Generalized Cauchy Integral Theorem)

Suppose that f is analytic interior to and on a simple closed positively oriented contour C. If z_0 is interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Proof. By induction. The base case ($n=0$) is just Cauchy's Integral formula. Let $n \geq 0$ and assume that the formula holds for n. We need to prove that

$$f^{(n+1)}(z_0) \overset{\text{def}}{=} \lim_{\Delta z \to 0} \frac{f^{(n)}(z_0+\Delta z) - f^{(n)}(z_0)}{\Delta z}$$

$$= \frac{(n+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{(n+1)+1}}\, dz.$$

Assume $|\Delta z|$ is so small that $z_0 + \Delta z$ is interior to C. Then by the inductive hypothesis

$$f^{(n)}(z_0+\Delta z) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0-\Delta z)^{n+1}}\, dz$$

and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Use the identity ($A, B \in \mathbb{C}$),

$$A^{n+1} - B^{n+1} = (A-B)(A^n + A^{n-1}B + \cdots + AB^{n-1} + B^n)$$

with $A = \dfrac{1}{z-z_0-\Delta z}$ and $B = \dfrac{1}{z-z_0}$. Then

$$\lim_{\Delta z\to 0}\frac{f^{(n)}(z_0+\Delta z)-f^{(n)}(z_0)}{\Delta z} = \lim_{\Delta z\to 0}\frac{n!}{2\pi i}\int_C \frac{f(z)}{\Delta z}\left(\frac{1}{(z-\Delta z-z_0)^{n+1}}-\frac{1}{(z-z_0)^{n+1}}\right)dz$$

$$= \lim_{\Delta z\to 0}\frac{n!}{2\pi i}\int_C \frac{f(z)}{\Delta z}\left(\frac{1}{z-z_0-\Delta z}-\frac{1}{z-z_0}\right)\left(A^n+A^{n-1}B+\cdots+B^n\right)dz$$

$$= \lim_{\Delta z\to 0}\frac{n!}{2\pi i}\int_C \frac{f(z)}{\Delta z}\left(\frac{z-z_0-(z-z_0-\Delta z)}{(z-z_0-\Delta z)(z-z_0)}\right)\left(\;\rule{2cm}{0.4pt}\;\right)dz$$

$$= \lim_{\Delta z\to 0}\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\left(A^n+A^{n-1}B+\cdots+B^n\right)dz$$

$$= \frac{n!}{2\pi i}\int_C \lim_{\Delta z\to 0}\frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\left(A^n+A^{n-1}B+\cdots+B^n\right)dz$$

$$= \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,\frac{(n+1)}{(z-z_0)^n}\,dz$$

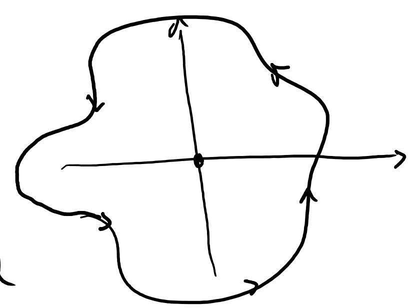
$$= \frac{(n+1)!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{(n+1)+1}}\,dz.$$

This completes the proof. ■

$\boxed{\text{Example (C.f. PSet 4 P5)}}$ Compute the integral

$$\frac{1}{2\pi i}\qquad \frac{(1+z)^n}{z^{k+1}}\,dz$$

where C is any simple closed positively oriented contour whose interior contains 0 and $0 \le k \le n$.



Let $f(z) = (1+z)^n$. Since f is entire, f is analytic inside and interior to C. Since 0 is interior to C, the generalized

Cauchy Integral formula applies:

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{k!}\left(\frac{k!}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz\right)$$

$$= \frac{1}{k!} f^{(k)}(0).$$

We have $f^{(k)}(z) = n(n-1)(n-2)\cdots(n-k+1)(1+z)^{n-k}$

$$\Rightarrow f^{(k)}(0) = n(n-1)(n-2)\cdots(n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

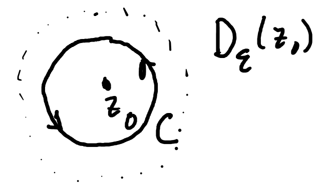
Hence, $\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,d = \frac{1}{k!}\frac{n!}{(n-k)!} = \binom{n}{k}.$

//

Theorem (Derivatives of Analytic Functions are Analytic)

Suppose that f is analytic at $z_0 \in \mathbb{C}$. Then for all $n \in \mathbb{N}$, $f^{(n)}$ is analytic at z_0.

Proof. Suppose f is analytic $z_0 \in \mathbb{C}$. Choose an open disk $D_q(z_0)$ on which f is analytic. We need to show that there is a neighborhood of z_0 where $f''(z)$ exists for all z in that neighborhood. Let C be the positively oriented circle of radius $\frac{q}{2}$ centered at z_0. Then f is analytic inside and interior to C, so by the generalized Cauchy Integral theorem,



$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds$$

for any z interior to C. Thus, f' has a derivative everywhere in the open set $D_{\varepsilon/2}(z_0)$. Thus, f' is analytic at z_0. By induction, $f^{(n)}$ is analytic for all $n \in \mathbb{N}$. ∎

$\boxed{\text{Corollary}}$ If $f(z) = u(x,y) + i\,v(x,y)$ is analytic at $z = x + iy$, then u and v have continuous partial derivatives of all orders at (x,y).

$\boxed{\text{Theorem (Morera's Theorem)}}$ Suppose f is continuous on a domain D. If

$$\int_C f(z)\,dz = 0$$

for every closed contour in D, then f is analytic on D.

Proof. By the Fundamental theorem of Contour Integrals, there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$. But F' is analytic on D by the preceding theorem. So f is analytic on D. □

When D is simply connected, Morera's theorem is just the converse of the Cauchy-Goursat theorem for simply connected domains

$\boxed{\text{Theorem (Cauchy's Inequalities)}}$ Suppose that f is analytic interior to and on a positively oriented circle $C_R(z_0)$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max_{z\in C_R(z_0)} |f(z)|.$$

Proof. By the generalized Cauchy Integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Hence,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

Triangle Ineq

$$\leq \frac{n!}{2\pi} \max_{z\in C_R(z_0)} \frac{|f(z)|}{|z-z_0|^{n+1}} \cdot 2\pi R$$

$$= \frac{n!}{2\pi} \cdot \max_{z\in C_R} |f(z)| \cdot \frac{2\pi R}{R^{n+1}}$$

$$= \frac{n!}{R^n} \max_{z\in C_R} |f(z)|. \qquad \blacksquare$$

Liouville's Theorem and Fundamental Theorem of Algebra

As an application, we will prove that every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$. In the language of algebra, this just means that $\mathbb{C}$ is algebraically closed. Thus, the theorem is "purely algebraic", although there is no "purely algebraic" proof.

The proof relies on the following theorem:

$\boxed{\text{Theorem (Liouville's Theorem)}}$ Every bounded entire function is constant.

Proof. The strategy is to show that $f'(z)=0$ for all $z \in \mathbb{C}$. This is sufficient to prove that f is constant on $\mathbb{C}$, since $\mathbb{C}$ is a domain. Let $z_0 \in \mathbb{C}$.

Since f is bounded, choose $M>0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $C_R(z_0)$ be a circle of radius $R>0$ centered at z_0. Then f is analytic inside and on $C_R(z_0)$, so by the Cauchy Inequality,

$$|f'(z_0)| \leq \frac{1!}{R} \max_{z \in C_R(z_0)} |f(z)| \leq \frac{M}{R} \xrightarrow{\text{as } R \to \infty} 0$$

Hence $|f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$. This proves the claim. ∎

$\boxed{\text{Theorem (Fundamental Theorem of Algebra)}}$ Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \quad a_1, \ldots, a_n \in \mathbb{C}$$

with degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Proof. Suppose to contrary that $p(z)$ has no root in $\mathbb{C}$. This means that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Hence $\frac{1}{p(z)}$ is entire. We show that $\frac{1}{p(z)}$ is bounded. By the lemma from week 1, choose $R>0$ such that

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n| R^n} \quad \text{for all } |z|>R.$$

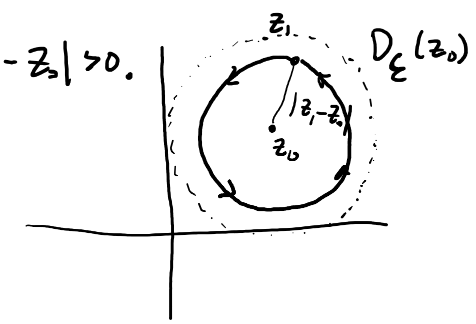
So $\frac{1}{p(z)}$ is bounded outside of the closed disk $\overline{D_R(0)}$. But $\overline{D_R(0)}$ is compact (closed and bounded) and $\frac{1}{p(z)}$ is continuous on $\overline{D_R(0)}$, so $\frac{1}{p(z)}$ is bounded $\overline{D_R(0)}$ by the extreme value theorem. Hence, $\frac{1}{p(z)}$ is bounded on $\mathbb{C}$. By Liouville's theorem, $\frac{1}{p(z)}$ is constant, say

$$\frac{1}{p(z)} = c, \quad \text{for some } c \in \mathbb{C}.$$

Then $p(z) = \frac{1}{c}$, a constant! This contradicts our assumption. ∎

Maximum Modulus Principle

Lemma Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in an analytic neighborhood $D_\varepsilon(z_0)$ of f. Then $f(z) = f(z_0)$ on $D_\varepsilon(z_0)$.



Proof. Let $z_1 \in D_\varepsilon(z_0) \setminus z_0$. Then set $\rho = |z_1 - z_0| > 0$. Let $C_\rho(z_0)$ be the circle of radius $\rho > 0$ centered at z_0. By the Cauchy Integral

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz \right|$$

$$= \frac{1}{2\pi} \left| \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz \right| \qquad z(t) = z_0 + \rho e^{it}, \quad t \in [0, 2\pi]$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}} \cdot \rho i e^{it}\, dt \right|$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt \right|$$

$$\underset{\text{T.I.}}{\leq} \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|$$

This proves

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt = |f(z_0)|.$$

Rewrite this

$$\int_0^{2\pi} |f(z_0 + \rho e^{it})| - |f(z_0)|\, dt = 0$$

Notice that $|f(z_0 + \rho e^{it})| - |f(z_0)| \geq 0$ for $t \in [0, 2\pi]$. The integrand is also continuous in t. Thus, we must have

$$|f(z_0 + \rho e^{it})| = |f(z_0)| \quad \text{on } [0, 2\pi].$$

Hence, $f(z) = f(z_0)$ for all $z \in C_\rho(z_0)$. By varying the radius ρ, we obtain $f(z) = f(z_0)$ for all $z \in D_\varepsilon(z_0)$. ∎

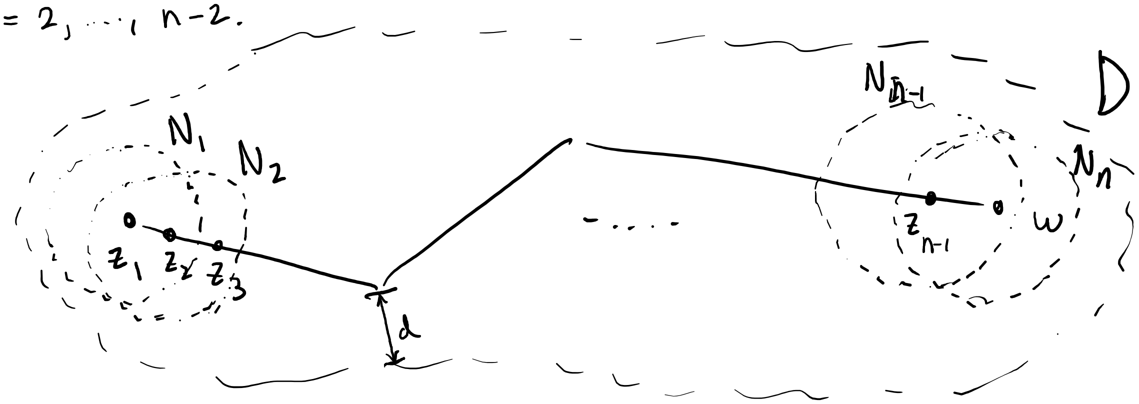
Theorem (Maximum Modulus Principle) Suppose that f is analytic and nonconstant on a domain D. Then $|f(z)|$ has no maximum value on D.

Proof. Suppose to the contrary that $|f(z)|$ has a maximum on D. Choose a point $z_0 \in D$ at which the maximum is reached. Let $w \in D$ be any other point and introduce a polygonal line L that joins z_0 to w.

We define a sequence of neighborhoods on which we can apply the lemma. Put $d = \min_{\substack{p \in L \\ q \in \partial D}} |p-q|$ unless $D = \mathbb{C}$, in which case take d to be any positive real number. We choose a finite sequence $z_1, \dots, z_n$ of points on L such that $z_0 = z_1$, $z_n = w$, and

$$|z_i - z_{i+1}| < d \qquad \text{for all } i = 1, \dots, n-1.$$

Let $N_i = D_d(z_i)$. By definition, $z_{i-1}, z_i, z_{i+1} \in D_d(z_i)$ for all $i = 2, \dots, n-2$.



Now, f achieves a maximum at $z_0 = z_1$ on D and hence also on N_1. Thus, $|f(z)| \le |f(z_1)|$ for all $z \in N_1$. By the lemma, $f(z) = f(z_1)$ on N_1. In particular, $f(z_0) = f(z_1) = f(z_2)$. But now f acheives a maximum on N_2 at z_2, so that $f(z_2) = f(z_3)$. Repeating this argument, we see that

$$f(z_0) = f(z_1) = f(z_2) = \cdots = f(z_n) = f(w).$$

Since w was arbitrary, we have shown that f is constant on D, contrary to hypothesis.

$\blacksquare$

Corollary Suppose f is continuous on a compact region R. If f is analytic and nonconstant on the interior of R, then f reaches a maximum only on the boundary.

Proof. By the Maximum Modulus Principle, f has no maximum on the interior of R. But R is compact, so f must achieve a maximum by the extreme value theorem. Therefore, the maximum occurs and it occurs only on the boundary.

$\blacksquare$